Rotations by Certain Special Angles

Bartłomiej BZDĘGA — Adam Mickiewicz University in Poznań

In issue no. 74 (Δ^2_{25}), I wrote about central symmetry, which is the same as a 180° rotation around the center of symmetry. This time, the topic will be rotations by 90° and 60°. The general rule for using rotations in problem-solving is as follows: we rotate a part of the diagram in such a way that it fits in a different place.

For precision – all figures are given here with the order of vertices counterclockwise, and we also perform rotations in this direction.

Example 1. In square $ABCD$, points P and Q lie on sides BC and CD, respectively, with $|\sphericalangle PAQ| = 45^\circ$ (Fig. 1). Prove that $|BP| + |DQ| = |PQ|$.

Solution. Rotate triangle ABP around point A by 90° – we get triangle ADP', which is congruent to ABP. The equalities $|\sphericalangle ADQ| = |\sphericalangle ADP'| = 90^\circ$ hold, so $|P'Q| = |DP'| + |DQ| = |BP| + |DQ|$. On the other hand, $|PQ| = |P'Q|$ because triangles APQ and AQP' are congruent (SSS).

A 60° rotation can be used to verify if a given triangle is equilateral. Triangle XYZ is equilateral if and only if one of the points X, Y, Z is the image of the other under a 60° rotation with respect to the third. An analogous rule can be formulated for the isosceles property of a right triangle, or even for any triangle.

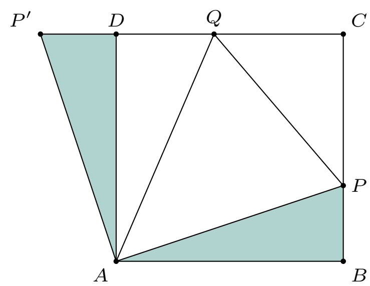

Fig. 1

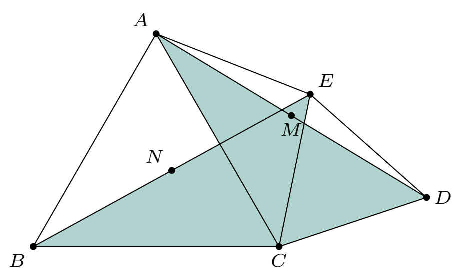

Fig. 2

Another general tip. If two congruent figures appear in a given geometric configuration, it's always worth checking what an isometry (for example, rotation) that transforms one into the other gives us.

Example 2. A convex pentagon $ABCDE$ is given, where triangles ABC and CDE are equilateral. Points M and N are the midpoints of diagonals AD and BE, respectively (Fig. 2). Prove that triangle CMN is equilateral.

Solution. Triangles ACD and BCE are congruent (SAS), the first is the image of the second under a 60° rotation around point C. The same rotation transforms point N into point M, so triangle CMN is equilateral.

Problems

1. On the sides of triangle ABC, squares $BPQC$ and $CRSA$ are constructed. Points K and L are the midpoints of segments BR and AQ, respectively. Prove that triangle CKL is an isosceles right triangle.
2. The common part of squares $ABCD$ and $APQR$ is point A. Point M is the midpoint of segment DP. Prove that $AM \perp BR$.
3. A point P is located inside square $A_1A_2A_3A_4$. Line ℓ_i passes through point A_i and is perpendicular to $A_{i+1}P$ for $i = 1, 2, 3, 4$ (where we take $A_5 = A_1$). Prove that lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4 intersect at a single point.
4. In triangle ABC, the median CM and the altitude CD are given. Through an arbitrary point P, lines perpendicular to AC, BC, and MC are drawn, intersecting line CD at points X, Y, and N, respectively. Prove that point N is the midpoint of segment XY.
5. A convex quadrilateral $ABCD$ is given. The perpendicular bisectors of segments AB and CD intersect at point P, where $|\sphericalangle APB| = |\sphericalangle CPD| = 120^\circ$. Prove that the midpoints of segments AB, BC, and CD determine an equilateral triangle.
6. Prove that all points X inside an equilateral triangle ABC that satisfy the equation $|AX|^2 + |BX|^2 = |CX|^2$ lie on a single circle.

Hints

1. Compare with Example 2.
2. Consider (congruent!) parallelograms $ADYP$ and $BARX$. A 90° rotation around the center of square $ABCD$ transforms the first one into the second, so the image of line AY under this rotation is line BR.
3. After a 90° rotation around the center of a given square, the line ℓ_i transforms into the line $A_{i+1}P$. Since after rotation all four lines have a common point (P), this must also have been the case before the rotation.
4. Mark the images of the points from the task as primed points under the 90° rotation. Triangle $X'Y'P$ is similar to triangle ABC, as it has corresponding sides parallel to those of triangle ABC. The line PN' contains the median of triangle $X'Y'P$ (why?), so point N' is the midpoint of segment $X'Y'$.
5. Triangle APC after a 120° rotation transforms into triangle BPD, so $|AC| = |BD|$ and the acute angle between them is 60°.
6. Let X be the point satisfying the conditions of the task. Rotate triangle ABX by 60° around point A – we obtain triangle ACX'. By the Pythagorean theorem, $|\sphericalangle CX'X| = 90^\circ$, so $|\sphericalangle AXB| = |\sphericalangle AX'C| = 150^\circ$.