$A(x)$ divides $B(x)$ as a polynomial (for that, consider $\gcd(A(x), B(x))$). Therefore $R(x) = -\frac{B(x)}{A(x)}$ is a polynomial, and hence $P(x, R(x)) = Q_1(x, R(x)) = 0$. $\square$

A careful reader will notice that Theorem 4 only gives R with rational, and not integer, coefficients. Indeed, if we took $P(x, y) = x^2 + x - 2y$, we would get $R(x) = \frac{x(x+1)}{2}$. We finish by presenting a problem illustrating that in some situations we can actually get $R(x)$ to have integer coefficients.

Problem. Suppose P, Q are two polynomials with integer coefficients, such that for any integer n we can find an integer m so that $P(n) = Q(m)$. Prove that then there is a polynomial $R(x)$ with rational coefficients such that $P(x) = Q(R(x))$. Moreover, if polynomial $Q(\frac{x}{k})$ does not have integer coefficients for any $k \geqslant 2$, prove that $R(x)$ has integer coefficients.

Can the policemen catch the thief?

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In this article, we will address the following problem.

In a village there is a thief and n policemen. The alleys in this village form an equilateral triangle along with its midlines (Fig. 1). The thief's maximum speed is $\kappa > 0$ times greater than the policemen's maximum speed. Given that everyone can see each other and the (continuous) movement is possible only along the alleys, determine whether the policemen can catch the thief from any starting position?

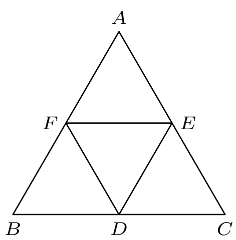

Fig. 1

Let us start the solution by analysing some simple cases. Assume that there is only one policeman, i.e. $n = 1$. This situation is trivial. If $\kappa < 1$ (i.e., the thief is slower than the policeman), the policeman has a winning strategy: he can chase the thief regardless of the route the latter takes, eventually catching him. If $\kappa \geqslant 1$, the thief has a winning strategy by looping in a cycle (such as $A \to B \to C \to A$ in Fig. 1) and adjusting his velocity and direction according to the policeman's movement.

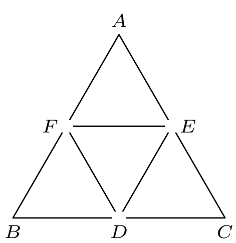

Fig. 2

The situation changes drastically (in favour of justice) if there are three (or more) policemen. We prove that in this case they will catch the thief regardless of his speed. One possible strategy for them is to position themselves at points D, E and F, respectively (using the notation of Fig. 1). In this way they partition the whole village into six connected parts (or components), as shown in Fig. 2. One of these parts contains the thief, and this part (like every other one) is closed at both ends by policemen. It is enough for one of them to move towards the other, thus catching the thief.

It remains to consider the case of two policemen, which is more demanding. First, we prove that for $\kappa \leqslant 3$ the police win. The first policeman is the one to chase the thief (that is why we'll call him *the chaser*). His only task is to prevent the thief from hiding forever in one place. More precisely, he has to make sure that the thief's *last-visited middle node* keeps changing over time (by *middle node* we mean D, E, or F). It is easy to see that only one policeman is enough to ensure this.

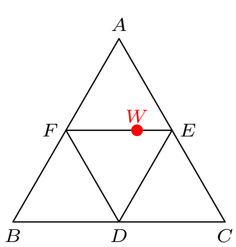

Fig. 3

The second policeman, whom we'll call *the watcher*), has a more subtle task. Firstly he needs to go to point W that splits the segment EF in a 1 : 2 ratio (Fig. 3); he does not need to hurry. When he gets there he needs to watch the thief closely (as watchers do). Basically his task is to cut off the thief's escape route whenever possible. For example, if the thief enters the 'upper corner' $E-A-F$ through point E, the watcher must prevent the thief from escaping this corner through point F. It is possible to do so since $\frac{|EA|+|AF|}{|WF|} = 3 \geqslant \kappa$. There is also one tiny catch – at the same time the thief comes back to the point E, the watcher needs to be back at point W. But this can be ensured if the watcher's velocity is always *exactly* three times smaller than the thief's (preserving an appropriate direction).

Here is the complete list of the watcher's guarding capabilities:

(a) if the thief enters $E{-}A{-}F$ through E, he cannot escape through F
(b) if the thief enters $E{-}A{-}F$ through F, he cannot escape through E
(c) if the thief enters $D{-}B{-}F$ through D, he cannot escape through F
(d) if the thief enters $D{-}C{-}E$ *or* $D{-}E$ through D, he cannot escape through E

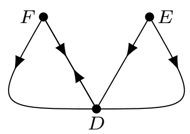

Fig. 4

As we already know, *the chaser* forces the thief to switch between nodes D, E, and F. Now, the conditions (a)–(d) impose strict restrictions on the order in which the thief can visit these nodes:

- only node D can be visited directly after E or F,
- only node F can be visited directly after D, and this can only be done via edge $D{-}F$.

These limitations are illustrated by a directed multigraph in Fig. 4. It turns out that the policemen are able to force the thief to run in the cycle $D \to F \to D$. But how can they eventually catch him? It is enough to provide one more instruction to the chaser: he must traverse the edges of the triangle DFB only clockwise. This does not change his ability to keep the thief moving and make him change the nodes, but in this way he will eventually bump into the thief when he is already forced to run in cycle. This completes the proof that the policemen have a winning strategy in this case.

Note that, contrary to his name, the chaser may be arbitrarily slow. But definitely he needs to be persistent

Now we prove that for $\kappa > 3$, it is the thief who has a winning strategy (with a proper initial placement). This is already suggested by the previous analysis. Basically his strategy is to start at one of the nodes D, E, F and move to another of those nodes whenever he is threatened by one of the policemen. Let us work out the details.

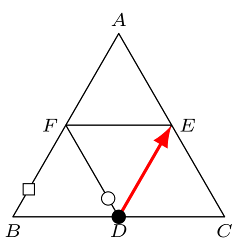

Fig. 5. The thief is depicted by a black circle, the policemen are white circle and square (the chaser is denoted by a circle)

Assume that the thief starts at node D. He stays there until one of the policemen (whom we call the chaser again) gets closer to him than, say, $\varepsilon = 0.2$ (we assume $|EF| = 1$). Then the thief attempts to move to either E or F. Without loss of generality assume that the chaser approaches the thief from the direction of points B or F. Assume that the thief covers length 1 in one minute. Consider the following cases:

(I) If the other policeman prevents the thief from going directly to F, then there is no policeman on edge $D{-}E$ and hence the thief can reach E in one minute – less than is needed for both policemen to get there (Fig. 5).

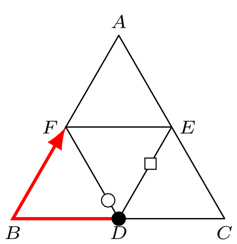

Fig. 6.

(II) If the other policeman is on edge $D{-}E$ then the thief can reach F in two minutes, and he gets there any policeman arrives (Fig. 6). Note that if the chaser is on segment BD, the thief can reach F in just one minute, but this is not important for our analysis.

(III) Otherwise the thief can directly reach E in 1 minute and F in at most 2 minutes. Note that it takes more than 2 minutes for the chaser (who initially is close to D) to get to any of the points E or F. As for the other policeman, Figure 7 presents two sets – light blue is the set of points, starting from which he can get to E in 1 minute and light green – the set of points, starting from which he can get to F in 2 minutes. It is clear that by the assumption $\kappa > 3$ those two sets do not intersect. Hence the thief can pick one of those nodes, being sure that he will reach it before the other policeman.

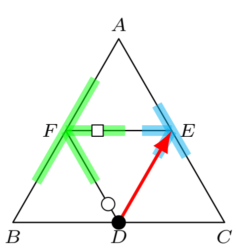

Fig. 7.

Of course, being able to switch nodes once implies the ability to do so indefinitely. Hence we can conclude that the thief has a winning strategy if and only if $n = 1$ and $\kappa \geqslant 1$ or $n = 2$ and $\kappa > 3$. Naturally, this problem can be generalized to other village configurations, such as the 2×2 grid or even more difficult ones, such as the $N \times M$ grid. We encourage the reader to explore similar strategies for these cases.