Markov Chains – part 1

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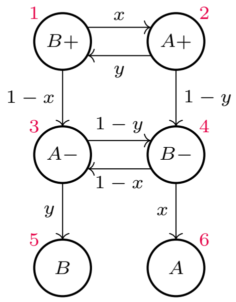
Let us consider the following problem:

Two players, A and B, are playing chess, with A starting. Each move can be either strong or weak. The first player to respond with a strong move to the opponent's weak move wins the game. Player A makes a strong move with probability x and a weak move with probability $1-x$. Similarly, player B has probabilities y and $1-y$. We assume that $0<x<1$ and $0<y<1$. For which values of x and y are the chances of winning for A and B equal?



Solution. Note that as long as nobody has won, the situation depends only on the last move. There are four possibilities: a strong/weak move by player A/B. Let us denote them by $A+$, $A-$, $B+$, and $B-$. The moment when player A starts can be considered separately, but there is no need for that because it is equivalent to the situation $B+$. So, we can take $B+$ as the starting point. Additionally, we include states A (indicating that A has won) and B (indicating that B has won). The diagram shows all six situations, with assigned numbers from 1 to 6, and connected by arrows representing transition probabilities.

We call such an object a finite *Markov chain.* It is described by a set of states $S_1, S_2, \ldots, S_n$ and transition probabilities $p_{i,j}$ for moving from state S_i to state S_j in one step for all $i, j \in 1, 2, \ldots, n$. This means that when in state S_i, the chain will transition to state S_1 with probability $p_{i,1}$, to state S_2 with probability $p_{i,2}$, and so on. For each i, the equation $p_{i,1}+p_{i,2}+\ldots+p_{i,n}=1$ holds. States S_i from which there are no outgoing transitions ($p_{i,i}=1$) are called *absorbing.*

We are interested in the probability of player A winning, which is the probability that the process ends in state S_6, *given* that it started in state S_1. To approach this problem, let q_i denote the probability that the process ends in state S_6, given that it started in state S_i. Note that $q_6=1$ and $q_5=0$. From state S_1, we can transition to S_2 (with probability x) or S_3 (with probability $1-x$). This implies that $q_1 = xq_2 + (1-x)q_3$. Similarly, we obtain the equations:

$$q_2 = yq_1 + (1-y)q_4, \quad q_3 = (1-y)q_4, \quad q_4 = (1-x)q_3 + x.$$

By solving this system of four equations with variables q_1, q_2, q_3, q_4, we obtain:

$$\tfrac{1}{2} = q_1 = \frac{x(1-y)}{(1-xy)(x+y-xy)} \iff x-y = xy(1-x)(1-y),$$

which means that for x and y satisfying the last equality, both players have an equal chance of winning. Note that it must hold that $x>y$, since the right-hand side of the last equality is positive.

Problems

1. The kitten wanders between the house, kindergarten, garden, field, and forest. It starts in the garden and always chooses one of the remaining four places. It always plays in the house and then goes to the field or back to the garden. After a walk in the forest, the kitten always goes either to the garden or to the kindergarten. If the kitten reaches the kindergarten, it never leaves from there (children, you know...). On the field, however, the kitten catches a mouse and ends its wanderings. Calculate the probability of ending up in the field. (We assume that the kitten's choices are random and equally likely.)
2. A student wants to buy their favorite energy drink, which costs 5 złotych. Unfortunately, the student has only 2 złote. So, they decide to go to a casino where, with a probability of p, they can win 3 złote for each złoty bet, or lose the bet with a probability of $1-p$. The student stops playing when they have enough money to buy the drink or when they run out of money. Depending on the value of p, determine the chances of the student achieving their goal. (Note: The author of this column does not endorse energy drinks or gambling in any way.)
3. Hansel and Gretel toss a fair coin. If the sequence HTT (where H denotes heads and T denotes tails) appears in three consecutive tosses, Hansel wins the game. If the sequence HHT appears, Gretel wins. Determine the probability of Hansel winning.
4. Seven children are standing in a circle, playing with a ball. Each child who currently has the ball throws it to the child standing immediately to their left (with probability $p < \frac{1}{2}$) or to the child standing immediately to their right (also with probability p), or takes the ball and goes back home (with probability $1-2p$). Depending on the value of p, calculate the probability that the same child who brought the ball home will return with it.

Hints and solutions

1. Let $S_1, S_2, \ldots, S_5$ denote the cat's presence at home, kindergarten, garden, field, and forest respectively. Let q_i denote the probability that the cat finishes its journey in the field (state S_4), given that it started in state S_i. Of course, $q_4=1$ and $q_2=0$. The value of q_3 is determined by solving the system of equations: $q_1 = \frac{1}{3}q_3 + \frac{1}{2}$, $q_3 = \frac{1}{4}q_1 + \frac{1}{4} + \frac{1}{4}q_5$, $q_5 = \frac{1}{2}q_3$.

2. State S_i for $i=0,1,2,3,4$ can be identified with the amount of money the student has, and state S_5 is the one in which the student has 5 or 6 złotys, i.e., enough to buy a drink.

3. In this problem, we can identify the states as follows: `start`, H and T (the outcome of the first toss and those following tails), HH, HT, HHT (Gretel's win), and HTT (Hansel's win). The probability of Hansel winning is $\frac{1}{3}$.
Fun fact. For every choice of a configuration of three consecutive outcomes resulting in Hansel's win, Gretel can choose a different configuration that gives her an advantage in such a game (*RROzważania O RReszce i OORle,* Δ^{11}_{15}).

4. Let us number the children from 1 to 7 in the order they stand in the circle. Let S_i be the state in which the i-th child has the ball, and let S_{7+i} be the state in which the i-th child takes the ball home. Assuming that the first child came with the ball, the initial state is S_1, and we are interested in q_8. Calculations are significantly simplified by the equalities $q_2=q_7$, $q_3=q_6$, and $q_4=q_5$, which follow from symmetry.