Club 44 F

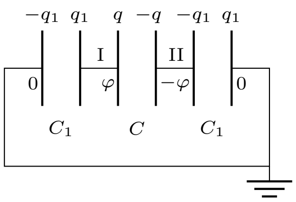
Edited by Elżbieta ZAWISTOWSKA

Solutions to problems from 3/2023

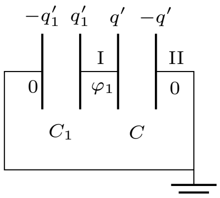
Let us recall problem statements:

754. The plates of a parallel plate capacitor with capacitance C are charged to potentials φ and $(-\varphi)$ relative to the ground. Each of the plates forms a capacitor with the ground, with capacitance C_1. Find the ratio of the electric field intensities between the plates of the capacitor with capacitance C initially and after grounding one of the plates.

755. A closed container is completely filled with water. Just above the bottom of the container, there is an air bubble. How will the pressure at the bottom change when the bubble rises to the surface?



Rys. 1



Rys. 2

754. The equivalent system to the one described in the problem is shown in Figure 1. A capacitor with capacitance C is charged with a charge of $q = 2\varphi C$, and the charges on the plates of capacitors with capacitances C_1 are $q_1 = \varphi C_1$. The total charge on the left plate is given by

(1) $$Q = q + q_1 = (2C + C_1)\varphi.$$

After grounding the right plate, the equivalent system is shown in Figure 2. The charge on the plates of the capacitor C is $q' = \varphi_1 C$, where φ_1 represents the potential of the ungrounded plate. The charge on the capacitor with capacitance C_1 is $q_1' = \varphi_1 C_1$, and the total charge on the left plate is

(2) $$Q' = q' + q_1' = \varphi(C + C_1) = Q.$$

Taking into account equation (1), we obtain

$$\varphi_1 = \frac{\varphi(2C + C_1)}{C + C_1}.$$

The electric field intensity between the capacitor plates is the ratio of the voltage to the distance between them. Therefore, the desired ratio of these intensities is

$$\frac{E}{E_1} = \frac{2\varphi}{\varphi_1} = \frac{2(C + C_1)}{2C + C_1}.$$

The rules of Club 44 F/M can be found on the webpage `deltami.edu.pl` (in Polish)

755. In the initial state, the air pressure inside the bubble is the same as the pressure of the water at the bottom of the container. The pressure difference between the bottom of the container and the top is given by $\Delta p = \rho g h$, where ρ is the density of water and h is the height of the container. During the ascent of the bubble, its volume remains unchanged because the liquid is practically incompressible. Therefore, the air pressure inside the bubble also remains constant. When the bubble reaches the top of the container, the pressure of the water under the surface is equal to the initial pressure at the bottom, so the pressure at the bottom has increased by Δp.

Club 44 M

Edited by Marcin E. KUCZMA

Solutions to problems from 3/2023

Let us recall problem statements:

857. Find all pairs of positive integers x, y such that $x^2 - 4y$ and $y^2 - 4x$ are squares of integers.

858. An equilateral triangle ABC of sidelength 1 and a segment DE of length 1 lie in the 3-space so that the segment has a point in common with the triangle. Show that one of the points A, B, C, D, E lies at a distance not exceeding 1 from each one of the other four points.

Problem 858 proposed by Michał Adamaszak of Copehagen.

857. Investigated are solutions of the system of equations

(1) $$x^2 = a^2 + 4y, \qquad y^2 = b^2 + 4x$$

in integers $x, y \geqslant 1$, $a, b \geqslant 0$. Obviously, x and a must be of equal parity; (and likewise y and b).

By symmetry it will suffice to consider $x \geqslant y$. Then $x^2 = a^2 + 4y \leqslant a^2 + 4x$, i.e.

(2) $$x^2 - 4x - a^2 \leqslant 0.$$

The first equation from (1) shows that $x^2 \geqslant a^2 + 4$, which (combined with the quadratic inequality (2)) yields the two-sided estimate

(3) $$\sqrt{a^2+4} \leqslant x \leqslant 2 + \sqrt{a^2+4}.$$

It is easily seen that the number $a + 2$ lies in the interval $\left[\sqrt{a^2+4},\, 2 + \sqrt{a^2+4}\right]$ (of length 2). If $a \geqslant 1$, this is the only integer of the same parity as a (in this interval); so $x = a + 2$. The system (1) now forces $y = a + 1$ and $(a+1)^2 = b^2 + 4(a+2)$. Rewrite the last equation as $(a-1)^2 = b^2 + 8$; i.e.,

$$(a - 1 - b)(a - 1 + b) = 8,$$

with the unique solution (in integers) $a = 4$, $b = 1$. Hence

$x = 6$, $y = 5$. The quadruple $(a, b, x, y) = (4, 1, 6, 5)$ satisfies the original system (1).

If, however, $a = 0$, the two-sided inequality (3) is fulfilled by two even integers $x = 2$ and $x = 4$. Plugging these into (1), the first of these two values yields contradiction, while the second one results in $(a, b, x, y) = (0, 0, 4, 4)$, which is a solution.

Taking symmetry into account (hence dismissing $x \geqslant y$), we obtain the following pairs (x, y): $(4, 4)$, $(6, 5)$ and $(5, 6)$.

858. This matrix can reveal the combinatorial nature of the problem:

$$\begin{bmatrix} i(A,D) & i(B,D) & i(C,D) \\ i(A,E) & i(B,E) & i(C,E) \end{bmatrix}$$

where

$$i(X,Y) = \begin{cases} 0 & \text{when } XY \leqslant 1 \\ 1 & \text{when } XY > 1 \end{cases}$$

(XY is the length of the segment with endpoints X, Y).

Since $AB = AC = BC = DE = 1$, the problem comes down to showing that at least one row or one column of this matrix has only zero entries. Suppose this is not the case. Then the following pattern appears in the matrix (up to a possible permutation of the symbols A, B, C and/or D, E, which does not influence the conditions of the problem): $\left[\begin{smallmatrix} 1 & 1 & \\ & & 1 \end{smallmatrix}\right]$; in terms of geometry (with notation as above):

$$AD > 1, \quad BD > 1, \quad CE > 1;$$

i.e.

$$AD > AC, \quad BD > BC, \quad CE > DE.$$

Let π be the perpendicular bisector plane of the segment CD. These inequalities imply that the points A and B lie on the same side of π as C, while E lies on the same side of π as D does (and no one of those points lies on π). Therefore π separates the triangle ABC from the segment DE, contradicting the condition that they meet. The result follows.

Are these distributed uniformly? ... that is on the $V/V_{\max}$ method

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In the XXI century, astronomers are obtaining *huge* amounts of observations with various level of specificity. Extracting knowledge from this *cosmos* of data obviously requires statistical analysis. Such analysis can be of different kinds: from extracting brightness and positions of objects from an image, through joint analysis of multiple observations in order to find a period of some phenomenon (e.g., eclipses in a stellar binary system), to analysis of data from various sources in order to determine parameters of a specific object (e.g., what is the distance to the Galactic center or what is the fraction of mass in the Universe that consists of barions).

Astronomy is significantly different from physics with respect to how the data to be analyzed are obtained: astronomy is based on observations of phenomena that we have no control of, whereas physics is mostly based on experiments performed (and hence controlled) by the scientist. This nature of astronomical observations poses a severe difficulty – sometimes increasing the sample of objects under study is extremely expensive and may require time that is longer than the expected lifetime of the researcher. Hence, astronomers often face incomplete samples of objects, even though they very much would like it to be otherwise. There are other obstacles, e.g., information about objects studied may come from observations taken under different conditions, epochs, etc. These subtle differences have to be taken into account during the statistical analysis, which is not an easy task.

In this article, I would like to present a statistical method that is typical for astronomy and bears an exotic name "$V/V_{\max}$." It was designed in the late 60s in order to tackle the following issue: we have a catalog of quasars with known brightness and redshift and we would like to know if quasars are distributed uniformly in space.

Quasar (from "quasi-stellar object") is a type of active galaxy that emits extremely bright radiation.

Note that for the most luminous quasars, the sample can be considered complete for very large distances, whereas for the less luminous quasars the sample is complete for smaller distances. At the first "glance" the quasar space density may seem to be getting lower with increasing distance from Earth due to different luminosities. However, this may be an artifact of higher overall completeness for smaller distances (compare Fig. 1 and Fig. 2).

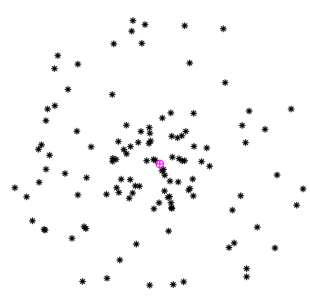

Fig. 1. Illustration of 150 quasars observed by a 2D astronomer in a 2D Universe. These quasars seem to show concentration around Earth (marked by the ⊕ symbol).

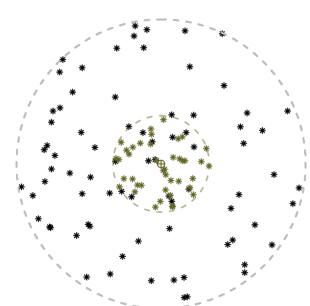

Fig. 2. Illustration of the same quasars as in Fig. 1 divided into two groups with different absolute magnitudes. Quasars from each group are uniformly placed in the area in which they can be seen.